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TRANSLATING A SUPPES-LEMMON STYLE NATURAL DEDUCTION INTO A SEQUENT CALCULUS

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ABSTRACT

This paper presents a straightforward procedure for translating a Suppes-Lemmon style natural deduction proof into an LK sequent calculus. In doing so, it illustrates a close connection between the two, and also provides an account of redundant steps in a natural deduction proof.

Keywords: natural deduction, sequent calculus

1. Introduction

This paper presents a method for a straightforward translation of a natural deduction proofs into a sequent-calculus derivation. In (Negri and von Plato 2001), the authors present a method for doing so for a Gentzen-style (Pelletier 1999) natural deduction, and the system used here is that in the more closely related (Restall 2014) style of Suppes (1957) and Lemmon (1965).

The translations here use the following method. Every line of a proof in natural deduction is an ordered quadruple $\langle L, (i), A, R \rangle$ where L is a list of premises the step depends on, (i) the line number, A a formula and R the justification, consisting of a rule of inference and the previous lines the step relies on. The list L consists of the numbers of lines where the premise has been introduced, but here they will be treated as the formula they stand for. A translation of a single line $\langle L, (i), A, R \rangle$ into a sequent calculus will yield a sequent $L \Rightarrow A$.

The translation procedure in this paper will construct a segment of a sequent-calculus derivation for every rule of inference of natural deduction, such that it will (1) end with (the sequent corresponding to) the line the rule produces and (2) begin with the (the sequent corresponding to) the line or lines that the rule of inference relies on. In such a manner, one will be able to construct a full derivation by starting with the conclusion of a proof and stacking segments corresponding to applications of rules one on top of the other. Some redundancy will occur, but that can be easily dealt with using proof-theoretic methods (primarily cut elimination).

2. Propositional

We begin with considering the rules of propositional logic. The sequent calculus used here will be *LK*, due to (Gentzen 1969), with some modifications due to (Ono 1998). In addition to axioms, the system consists of the structural rules (weakening, contraction, exchange, cut):

1. $\dfrac{\Gamma \Longrightarrow \Delta}{A, \Gamma \Longrightarrow \Delta}\,\mathbf{L}W \qquad \dfrac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, A}\,\mathbf{L}W$

2. $\dfrac{A, A, \Gamma \Longrightarrow \Delta}{A, \Gamma \Longrightarrow \Delta}\,\mathbf{L}C \qquad \dfrac{\Gamma \Longrightarrow \Delta, A, A}{\Gamma \Longrightarrow \Delta, A}\,\mathbf{R}C$

3. $\dfrac{\Gamma', A, B, \Gamma \Longrightarrow \Delta}{\Gamma', B, A, \Gamma \Longrightarrow \Delta}\,\mathbf{L}E \qquad \dfrac{\Gamma \Longrightarrow \Delta, A, B, \Delta'}{\Gamma \Longrightarrow \Delta, B, A, \Delta'}\,\mathbf{R}E$

4. $\dfrac{\Gamma \Rightarrow \Theta, A \qquad A, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\,\mathbf{Cut}$

Moreover, the rules for propositional symbols are:

1. $\dfrac{\Gamma \Longrightarrow \Delta, A}{\neg A, \Gamma \Longrightarrow \Delta}\,(\mathbf{L}\neg) \qquad \dfrac{A, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg A}\,(\mathbf{R}\neg)$

2. $\dfrac{A, \Gamma \Longrightarrow \Delta}{A \wedge B, \Gamma \Longrightarrow \Delta}\,(\mathbf{L}\wedge)^* \qquad \dfrac{\Gamma \Longrightarrow \Delta, A \qquad \Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \wedge B}\,(\mathbf{R}\wedge)$

3. $\dfrac{A, \Gamma \Longrightarrow \Delta \qquad B, \Gamma \Longrightarrow \Delta}{A \vee B, \Gamma \Longrightarrow \Delta}\,(\mathbf{L}\vee) \qquad \dfrac{\Gamma \Longrightarrow \Delta, A}{\Gamma \Longrightarrow \Delta, A \vee B}\,(\mathbf{R}\vee)^*$

4. $$\frac{B,\Gamma \Longrightarrow \Delta \qquad \Gamma \Longrightarrow \Delta, A}{A \to B, \Gamma \Longrightarrow \Delta}\ (\mathbf{L}\to) \qquad \frac{A,\Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \to B}\ (\mathbf{R}\to)$$

* - the rules L∧ and R∨ can also, respectively, produce the formula $B \wedge A$ and $B \vee A$.

We now proceed to give a translation for each of the propositional rules of natural deduction. In the derivations below, the sequents corresponding to lines of a proof will be marked by their line number. This is simply to make keeping track of them easier, and is not part of the actual derivation.

2.1. Premise

The sequent corresponding to the application of the premise rule is $A \Rightarrow A$. That any such sequent is derivable is readily apparent.

2.2. Negation

The Introduction rule, ¬I, has the following form:

i	(i)	A	Pr.
i,Γ	(j)	$B \wedge \neg B$	
Γ	(k)	$\neg A$	¬I, i, j

This is transformed into:

$$\dfrac{\dfrac{(i)\, A \Rightarrow A \qquad \dfrac{(j)\, A,\Gamma \Rightarrow B \wedge \neg B \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{B, \neg B \Rightarrow}\ \mathbf{L}\neg}{B \wedge \neg B, \neg B \Rightarrow}\ \mathbf{L}\wedge}{\neg B, B \wedge \neg B \Rightarrow}\ \mathbf{L}E}{B \wedge \neg B, B \wedge \neg B \Rightarrow}\ \mathbf{L}\wedge}{B \wedge \neg B \Rightarrow}\ \mathbf{L}C}{A,\Gamma \Rightarrow}\ \mathbf{Cut}}{A,\Gamma \Rightarrow}\ \mathbf{Cut}}{(k)\, \Gamma \Rightarrow \neg A}\ \mathbf{R}\neg$$

The Elimination rule, ¬E, has the following form:

Γ	(i)	$\neg\neg A$	
Γ	(j)	A	¬E, i

This is transformed into:

$$\dfrac{(i)\,\Gamma \Rightarrow \neg\neg A \qquad \dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}\,\mathbf{R}\neg}{\neg\neg A \Rightarrow A}\,\mathbf{L}\neg}{(j)\,\Gamma \Rightarrow A}\,\mathbf{Cut}$$

2.3. Conjunction

The Introduction rule, ∧I, has the following form:

Γ_1	(i)	A	
Γ_2	(j)	B	
Γ_1,Γ_2	(k)	$A \wedge B$	∧I, i, j

This is translated into the following segment:

$$\dfrac{\dfrac{(i)\,\Gamma_1 \Rightarrow A}{\Gamma_1,\Gamma_2 \Rightarrow A}\,\mathbf{L}W, \mathbf{L}E \qquad \dfrac{(j)\,\Gamma_2 \Rightarrow B}{\Gamma_1,\Gamma_2 \Rightarrow B}\,\mathbf{L}W}{(k)\,\Gamma_1,\Gamma_2 \Rightarrow A \wedge B}\,\mathbf{R}\wedge$$

The Elimination rule, ∧E, has the following form:

Γ	(i)	$A \wedge B$	
Γ	(j)	A	∧E, i

This is transformed into:

$$\dfrac{(i)\,\Gamma \Rightarrow A \wedge B \qquad \dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}\,\mathbf{L}\wedge}{(j)\,\Gamma \Rightarrow A}\,\mathbf{Cut}$$

2.4. Disjunction

The Introduction rule, ∨I, has the following form:

Γ	(i)	A
Γ	(j)	$A \vee B$

This is straightforwardly transformed into:

$$\dfrac{(i)\,\Gamma \Rightarrow A}{(j)\,\Gamma \Rightarrow A \vee B}\,\mathbf{R}\vee$$

The Elimination rule, ∨E, has the following form:

Γ_1	(i)	$A \vee B$	
j	(j)	A	Pr
j,Γ_2	(k)	C	
l	(l)	B	Pr.
l,Γ_3	(m)	C	
$\Gamma_1,\Gamma_2,\Gamma_3$	(n)	C	∨E, i, j, k, l, m

This is transformed into (the derivation broken into two parts for legibility):

$$\dfrac{\dfrac{\dfrac{(j)\,A \Rightarrow A \qquad (k)\,A,\Gamma_2 \Rightarrow C}{A,\Gamma_2 \Rightarrow C}\,\textbf{Cut}}{A,\Gamma_2,\Gamma_3 \Rightarrow C}\,\textbf{L}W,\textbf{L}E \qquad \dfrac{\dfrac{(l)\,B \Rightarrow B \qquad (m)\,B,\Gamma_3 \Rightarrow C}{B,\Gamma_3 \Rightarrow C}\,\textbf{Cut}}{B,\Gamma_2,\Gamma_3 \Rightarrow C}\,\textbf{L}W,\textbf{L}E}{A \vee B,\Gamma_2,\Gamma_3 \Rightarrow C}\,\textbf{L}\vee$$

We now use this segment above the right upper sequent of the following one to obtain the full segment corresponding to the rule:

$$\dfrac{(i)\,\Gamma_1 \Rightarrow A \vee B \qquad A \vee B,\Gamma_2,\Gamma_3 \Rightarrow C}{(n)\,\Gamma_1,\Gamma_2,\Gamma_3 \Rightarrow C}\,\textbf{Cut}$$

2.5. Implication

The Introduction rule, →I, has the following form:

i	(i)	A	Pr.
i,Γ	(j)	B	
Γ	(k)	$A \to B$	→I, i, j

This is transformed into:

$$\dfrac{\dfrac{(i)\,A \Rightarrow A \qquad (j)\,A,\Gamma \Rightarrow B}{A,\Gamma \Rightarrow B}\,\textbf{Cut}}{(k)\,\Gamma \Rightarrow A \to B}\,\textbf{R}\to$$

The Elimination rule, →E, has the following form:

Γ_1	(i)	$A \to B$	
Γ_2	(j)	A	
Γ_1,Γ_2	(k)	B	→E, i, j

This is transformed into:

$$\dfrac{(i)\,\Gamma_1 \Rightarrow A \to B \qquad \dfrac{\dfrac{B \to B}{B,\Gamma_2 \Rightarrow B}\,\textbf{L}W,\textbf{L}E \qquad \dfrac{(j)\,\Gamma_2 \Rightarrow A}{\Gamma_2 \Rightarrow B,A}\,\textbf{R}W,\textbf{R}E}{A \to B,\Gamma_2 \Rightarrow B}\,\textbf{L}\to}{(k)\,\Gamma_1,\Gamma_2 \Rightarrow B}\,\textbf{Cut}$$

3. Quantification

Here, we will add the required rules of *LK*:

1. $\dfrac{A[t/x],\Gamma \Longrightarrow \Delta}{\forall xA,\Gamma \Longrightarrow \Delta}\,\textbf{L}\forall \qquad \dfrac{\Gamma \Longrightarrow \Delta,A[t/x]}{\Gamma \Longrightarrow \Delta,\forall xA}\,\textbf{R}\forall^*$

2. $$\frac{A[t/x], \Gamma \Longrightarrow \Delta}{\exists x A, \Gamma \Longrightarrow \Delta}\ \mathbf{L}\exists^* \qquad \frac{\Gamma \Longrightarrow \Delta, A[t/x]}{\Gamma \Longrightarrow \Delta, \exists x A}\ \mathbf{R}\exists$$

* - the constant t does not occur anywhere in Γ, Δ or A.

We now proceed to quantified derivations, starting with the universal quantifier.

3.1. Universal

The Introduction rule, ∀I, has the following form:

Γ	(i)	$A[t/x]$	
Γ	(j)	$\forall x A$	∀I, i

where t does not appear in Γ or A.

This is straightforwardly transformed into:

$$\frac{(i)\ \Gamma \Rightarrow A[t/x]}{(j)\ \Gamma \Rightarrow \forall x A}\ \mathbf{R}\forall$$

The Elimination rule, ∀E, has the following form:

Γ	(i)	$\forall x A$	
Γ	(j)	$A[t/x]$	∀E, i

This is transformed into:

$$\frac{(i)\ \Gamma \Rightarrow \forall x A \qquad \dfrac{A[t/x] \Rightarrow A[t/x]}{\forall x A \Rightarrow A[t/x]}\ \mathbf{L}\forall}{(j)\ \Gamma \Rightarrow A[t/x]}\ \mathbf{Cut}$$

3.2. Existential

The Introduction rule, ∃I, has the following form:

Γ	(i)	$A[t/x]$	
Γ	(j)	$\exists x A$	∃I, i

This is, again straightforwardly, transformed into:

$$\frac{(i)\ \Gamma \Rightarrow A[t/x]}{(j)\ \Gamma \Rightarrow \exists x A}\ \mathbf{R}\exists$$

The Elimination rule, ∃E, has the following form:

$$\begin{array}{rlll} \Gamma_1 & \text{(i)} & \exists x A & \\ j & \text{(j)} & A[t/x] & \text{Pr.} \\ j,\Gamma_2 & \text{(k)} & B & \\ \Gamma_1,\Gamma_2 & \text{(l)} & B & \exists\text{E, i, j, k} \end{array}$$

where t does not appear in Γ_1, Γ_2, A or B.

This is transformed into:

$$\dfrac{(i)\,\Gamma_1 \Rightarrow \exists x A \qquad \dfrac{\dfrac{(j)\,A[t/x] \Rightarrow A[t/x] \qquad (k)\,A[t/x],\Gamma_2 \Rightarrow B}{A[t/x],\Gamma_2 \Rightarrow B}\,\textbf{Cut}}{\exists x A,\Gamma_2 \Rightarrow B}\,\textbf{L}\exists}{(l)\,\Gamma_1,\Gamma_2 \Rightarrow B}\,\textbf{Cut}$$

4. Identity

For identity we add two more rules to our sequent calculus, due to (Negri and von Plato 2001).

$$\dfrac{a=a,\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textbf{Ref} \qquad \dfrac{A[b/x],a=b,A[a/x],\Gamma \Rightarrow \Delta}{a=b,A[a/x],\Gamma \Rightarrow \Delta}\,\textbf{Repl}$$

The Introduction rule, =I, has the following form:

$$\text{(i)} \quad a=a \quad =\text{I}$$

This is transformed into:

$$\dfrac{a=a \Rightarrow a=a}{(i)\ \Rightarrow a=a}\,\textbf{Ref}$$

The Elimination rule, =E, has the following form:

$$\begin{array}{rlll} \Gamma_1 & \text{(i)} & A[b/x] & \\ \Gamma_2 & \text{(j)} & a=b & \\ \Gamma_1,\Gamma_2 & \text{(k)} & A[a/x] & =\text{E, i, j} \end{array}$$

Before proceeding, we will prove two simple lemmas:

Lemma 4.1. $a=b \Rightarrow b=a$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{b=a \Rightarrow b=a}{b=a,a=b,a=a \Rightarrow b=a}\,\textbf{L}W,\textbf{L}E}{a=b,a=a \Rightarrow b=a}\,\textbf{Repl}}{a=a,a=b \Rightarrow b=a}\,\textbf{L}E}{a=b \Rightarrow b=a}\,\textbf{Ref}$$

□

Now, using this lemma we prove the following:

Lemma 4.2. $a = b, A[b/x] \Rightarrow A[a/x]$

Proof.

$$\dfrac{\dfrac{}{a=b\Rightarrow b=a}\,\textbf{Lemma 4.1} \qquad \dfrac{\dfrac{A[a/x]\Rightarrow A[a/x]}{A[a/x],b=a,A[b/x]\Rightarrow A[a/x]}\,\textbf{L}W\textbf{, L}E}{b=a,A[b/x]\Rightarrow A[a/x]}\,\textbf{Repl}}{a=b,A[b/x]\Rightarrow A[a/x]}\,\textbf{Cut}$$

□

And finally, the transformation for the rule =E is (with the two simple lemmas in their respective places):

$$\dfrac{(i)\,\Gamma_1\Rightarrow A[b/x] \qquad \dfrac{\dfrac{(j)\,\Gamma_2\Rightarrow a=b \qquad \dfrac{}{a=b,A[b/x]\Rightarrow A[a/x]}\,\textbf{Lemma 4.2}}{\Gamma_2,A[b/x]\Rightarrow A[a/x]}\,\textbf{Cut}}{A[b/x],\Gamma_2\Rightarrow A[a/x]}\,\textbf{L}E}{(k)\,\Gamma_1,\Gamma_2\Rightarrow A[a/x]}\,\textbf{Cut}$$

5. Concluding Remarks

We can now easily see that any proof of natural deuction can be transformed into a full sequent calculus derivation: any proof will begin with either the application of a Premise or =I rule, each of which can be translated. Moreover, it will proceed through a finite number of step, each in line with some rule of inference, any of which are likewise translatable. Therefore, any proof is translatable.

5.1. Redundancy

It is clear a number of steps in the derivation are redundant. First and foremost, whenever a rule calls for an assumption to be introduced (e.g. ¬I or →I), there is an application of cut in which one of the upper sequents and the lower sequent are the same. Obviously, these instances can be eliminated. However, they are retained so that every line listed in the justification would be present. In such a way, one can account for a line being redundant in a natural deduction proof, namely, when its corresponding sequent does not occur in a sequent-calculus derivation, translated in this manner. Consider the following example:

1	(1)	$A \wedge B$	Pr.
1	(2)	A	∧E, 1
1	(3)	B	∧E, 1
	(4)	$(A \wedge B) \rightarrow A$	→I, 1, 2

The translation of this proof would be (remember we build derivations starting from the conclusion):

$$\dfrac{\dfrac{(1)\, A \wedge B \Rightarrow A \wedge B \qquad \dfrac{(1)\, A \wedge B \Rightarrow A \wedge B \qquad \dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}\ \mathbf{L}\wedge}{(2)\, A \wedge B \Rightarrow A}\ \mathbf{Cut}}{A \wedge B \Rightarrow A}\ \mathbf{Cut}}{(4)\ \Rightarrow (A \wedge B) \rightarrow A}\ \mathbf{R}\rightarrow$$

Clearly, line (3) of the proof is redundant, and translating it into a sequent calculus in the manner suggested in this paper clearly shows it.

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